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UNSTEADY MOTION OF A VISCOUS LIQUID BETWEEN ROTATING PARALLEL WALLS IN THE PRESENCE OF A CROSSFLOW

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The paper considers the unsteady flow of a viscous incompressible fluid inside an infinitely long slot with uniform injection or suction of the fluid through the porous walls of the slot. The plates with the fluid are rotated rigidly with constant angular velocity. The unsteady flow is induced by nontorsional vibrations of the upper plate. The flow-velocity field and the tangential stress vectors exerted by the fluid on the upper and lower walls of the slot are determined. In this case, one can find an exact solution of the three-dimensional nonstationary Navier–Stokes equations. No restrictions are imposed on the motion pattern of the plate.

The nonstationary problem of a boundary layer on a rotating plate in the absence of injection was considered previously [1]. The nonstationary problem of a half-space bounded by a porous plate in the presence of injection (suction) of the medium was solved in [2].

In the present paper, we study the unsteady flow of a homogeneous incompressible fluid. The slot is formed by two infinite parallel porous plates Q_1 and Q_2 which are l apart. The fluid is in a mass force field with potential U. The plates with the viscous fluid rotate in the space uniformly and rigidly with constant angular velocity $\boldsymbol{\omega}_0$. The vector $\boldsymbol{\omega}_0$ makes constant angle α $(0 < \alpha \leqslant \pi/2)$ with the planes of the plates.

At the initial time, the upper porous plate Q_1 begins to move with specified velocity $\boldsymbol{u}(t)$. The lower plate remains fixed in a moving coordinate system. At the same moment, the fluid is injected (sucked off) through the upper plate with velocity $\boldsymbol{u}_0(t)$ along the normal to the plate surface. We attach a Cartesian coordinate system $Oxyz$ with unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ to the upper plane Q_1 of the slots, so that the plane Oxz coincides with the plane Q_1 and the y axis is directed perpendicular to Q_1 into the depth of the fluid.

We assume that at the initial time $t = 0$, the upper porous plate, through which the fluid is injected or sucked off with velocity $\boldsymbol{u}_0(t)$ along the normal to it, begins to move in a longitudinal direction with velocity $\boldsymbol{u}(t)$.

The fluid flow is described by the Navier–Stokes equations and the following boundary and initial conditions in the system $Oxyz$ rotating with angular velocity $\boldsymbol{\omega}_0$:

$$\boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \boldsymbol{r}) + 2\boldsymbol{\omega}_0 \times \boldsymbol{V} + \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}(\nabla \boldsymbol{V}) = -\frac{1}{\rho}\nabla P + \nabla U + \nu \Delta \boldsymbol{V},$$

$$\operatorname{div} \boldsymbol{V} = 0 \ \text{ in slot}, \qquad \boldsymbol{V}\Big|_{Q_1} = \{\boldsymbol{u}(t), \boldsymbol{u}_0(t)\}, \qquad \boldsymbol{V}\Big|_{Q_2} = \boldsymbol{u}_0(t), \qquad t > 0. \tag{1}$$

Here t is time, $\boldsymbol{V}$ is the fluid velocity, P is the pressure, ρ is the density, and ν is the kinematic viscosity. Motion of the fluid begins from the state of rest: $\boldsymbol{V}(0, \boldsymbol{r}) = 0$.

A solution of system (1) is sought in the form

$$\boldsymbol{V} = \{V_x(y,t), u_0(t), V_z(y,t)\},$$

$$P = \frac{1}{2}\rho(\boldsymbol{\omega}_0 \times \boldsymbol{r})^2 + \rho U + \rho x 2\omega_{0z} u_0(t) - \rho z 2\omega_{0x} u_0(t) - \rho y \frac{\partial u_0(t)}{\partial t} + \rho s(y,t),$$

where $s(y,t)$ is the pressure.

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For the velocity and pressure fields, we have the system

$$\frac{\partial V_x}{\partial t} + 2\omega_{0y} V_z = LV_x, \qquad \frac{\partial V_z}{\partial t} - 2\omega_{0y} V_x = LV_z,$$
$$\frac{\partial s}{\partial y} = 2(\omega_{0z} V_x - \omega_{0x} V_z), \qquad 0 \leqslant y \leqslant l, \tag{2}$$

where $L = \nu \partial^2/\partial y^2 - u_0(t)\partial/\partial y$.

A solution of system (2) is sought in the form

$$\boldsymbol{V} = \boldsymbol{W} \sin 2\Omega t - (\boldsymbol{W} \times \boldsymbol{e}_y) \cos 2\Omega t, \tag{3}$$

where $\boldsymbol{W}(y, t)$ is a new unknown function $\Omega = \omega_{0y}$. The function $\boldsymbol{W}$ satisfies a parabolic equation and the boundary conditions

$$\frac{\partial \boldsymbol{W}}{\partial t} = L\boldsymbol{W}, \qquad 0 \leqslant y \leqslant l, \tag{4}$$

where $\boldsymbol{W}(0, t) = \boldsymbol{u}(t) \sin 2\Omega t + \boldsymbol{u}(t) \times \boldsymbol{e}_y \cos 2\Omega t$, $t > 0$, $\boldsymbol{W}(y, 0) = 0$, and $\boldsymbol{W}(l, t) = 0$.

We consider the case where $\boldsymbol{u}_0(t) = a = \text{const}$, which corresponds to uniform injection or suction. In this case, $a > 0$ corresponds to injection of the medium through the upper wall of the slot and $a < 0$ corresponds to suction.

Using the Duhamel integral, we write the solution of problem (4) in the form

$$\boldsymbol{W}(y, t) = \frac{\partial}{\partial t} \int\limits_0^t \boldsymbol{W}(0, t - \tau) W_1(y, \tau)\, d\tau. \tag{5}$$

Here $W_1(y, t)$ is a solution of the boundary-value problem

$$\frac{\partial W_1}{\partial t} + a\, \frac{\partial W_1}{\partial y} = \nu\, \frac{\partial^2 W_1}{\partial y^2}, \tag{6}$$

$$W_1(0, t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases} \qquad W_1(l, t) = 0.$$

To solve Eq. (6), we use operational calculus. We introduce a Laplace transform of a function by the relation

$$\tilde{u}(y, p) = \int\limits_0^\infty \exp(-pt) u(y, t)\, dt.$$

In the space of transforms, Eq. (6) becomes

$$\nu\, \frac{\partial^2 W_1}{\partial y^2}\,(y, p) - a\, \frac{\partial \tilde{W}_1}{\partial y}\,(y, p) - p\tilde{W}_1(y, p) = 0, \tag{7}$$

where

$$\tilde{W}_1(0, p) = 1/p, \qquad \tilde{W}_1(l, p) = 0. \tag{8}$$

The solution of Eq. (7) has the form

$$\tilde{W}_1(y, p) = C_1 \exp(\lambda_1 y) + C_2 \exp(\lambda_2 y). \tag{9}$$

Determining the constants C_1 and C_2 from boundary conditions (8), we transform solution (9):

$$\tilde{W}_1(y, p) = \frac{1}{p} \exp(\mu y)\, \frac{\sinh[(l - y)\sqrt{p/\nu + \mu^2}\,]}{\sinh(l\sqrt{p/\nu + \mu^2}\,)}, \qquad \mu = \frac{a}{2\nu}. \tag{10}$$

We denote $q = \sqrt{p/\nu + \mu^2}$ and take partial fractions of $\psi = \sinh[(l - y)q]/\sinh(lq)$:

$$\psi = 1 - \frac{y}{l} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{(-1)^n}{n}\, \frac{q^2}{q^2 + (\pi n/l)^2} \sin \pi n(1 - y/l)$$
$$= 1 - \frac{y}{l} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{(-1)^n}{n}\, \frac{p + \mu^2\nu}{p + \mu^2\nu + (\pi n/l)^2\nu} \sin \pi n(1 - y/l). \tag{11}$$

We denote $\lambda_n = \pi n/l$. Then, according to the well-known formulas of operational calculus [3], we obtain

$$L^{-1}\Big(\frac{p+\mu^2\nu}{P+\mu^2\nu+(\pi n/l)^2\nu}\Big) = \frac{\mu^2+\lambda_n^2\exp[-(\mu^2+\lambda_n^2)\nu t]}{\mu^2+\lambda_n^2}, \tag{12}$$

where L^{-1} is the inverse Laplacian.

Substituting (11) into (10) with allowance for (12), we obtain the following solution of Eqs. (6) in the space of originals:

$$W_1(y,t) = \exp(\mu y)\Bigg(1-\frac{y}{l}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\lambda_n y\,\frac{\mu^2+\lambda_n^2\exp[-\nu(\mu^2+\lambda_n^2)t]}{\mu^2+\lambda_n^2}\Bigg). \tag{13}$$

Thus, the solution of problem (4) is defined by formulas (5) and (13).

Substituting (5) into (3) with allowance for (13), we obtain the required velocity field:

$$\boldsymbol{V} = \sin(2\Omega t)\,\frac{\partial}{\partial t}\int_0^t \boldsymbol{W}(0,t-\tau)W_1(y,t)\,d\tau + \cos(2\Omega t)\boldsymbol{e}_y\times\frac{\partial}{\partial t}\int_0^t \boldsymbol{W}(0,t-\tau)W_1(y,t)\,d\tau. \tag{14}$$

After transformations, formula (14) is brought to the form

$$\boldsymbol{V} = \boldsymbol{T}(0,t)W_1(y,t) + \int_0^t \{[\boldsymbol{u}(t-\tau) - 2\Omega\boldsymbol{u}(t-\tau)\times\boldsymbol{e}_y]\cos 2\Omega\tau$$

$$+\,[\dot{\boldsymbol{u}}(t-\tau)\times\boldsymbol{e}_y + 2\Omega\boldsymbol{u}(t-\tau)]\sin 2\Omega\tau\}W_1(y,\tau)\,d\tau,$$

where $\boldsymbol{T}(0,t) = \boldsymbol{u}(0)\cos 2\Omega t + \boldsymbol{u}(0)\times\boldsymbol{e}_y\sin 2\Omega t$.

Further transformations are conveniently performed in complex form. We introduce the complex vectors $\hat{V} = V_x + iV_z$ and $\hat{u} = u_x + iu_z$. Then, $\boldsymbol{u}\times\boldsymbol{e}_y = i\hat{u}$, $\boldsymbol{W}(0,t) = iu(t)\exp(-2i\Omega t)$, and $\boldsymbol{T}(0,t) = \boldsymbol{u}(0)\exp(2i\Omega t)$.

Formulas (5) and (13) are written as

$$\hat{W} = i\,\frac{\partial}{\partial t}\int_0^t \hat{u}(\tau)\exp(-2i\Omega\tau)W_1(y,t-\tau)\,d\tau,$$

$$\hat{V} = \exp(2i\Omega t)\,\frac{\partial}{\partial t}\int_0^t \hat{u}(\tau)\exp(-2i\Omega\tau)W_1(y,t-\tau)\,d\tau,$$

where

$$W_1(y,t) = \exp(\mu y)\Bigg(1-\frac{y}{l}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{\mu^2+\lambda_n^2\exp[-\nu(\mu^2+\lambda_n^2)t]}{\mu^2+\lambda_n^2}\sin\lambda_n y\Bigg).$$

The tangential stress vectors exerted by the fluid on the upper and lower walls of the slot are obtained from the formulas

$$\hat{f}_0 = \rho\nu\,\frac{\partial\hat{V}}{\partial y}\Big|_{y=0}, \qquad \hat{f}_l = \rho\nu\,\frac{\partial\hat{V}}{\partial y}\Big|_{y=l}.$$

We finally obtain

$$\hat{f}_0 = \exp(2i\Omega t)\,\frac{\partial}{\partial t}\int_0^t \hat{u}(\tau)\exp(-2i\Omega\tau)\,\frac{\partial u}{\partial y}(0,t-\tau)\,d\tau,$$

$$\frac{\partial u}{\partial y}\Big|_{y=0} = \mu(1-\coth\mu l) - \frac{2}{l}\sum_{n=1}^{\infty}\frac{\lambda_n^2\exp[-(\mu^2+\lambda_n^2)\nu t]}{\mu^2+\lambda_n^2},$$

$$\hat{f}_l = \exp(2i\Omega t)\,\frac{\partial}{\partial t}\int_0^t \hat{u}(\tau)\exp(-2i\Omega\tau)\,\frac{\partial u}{\partial y}(l,t-\tau)\,d\tau,$$

$$\frac{\partial u}{\partial y}\Big|_{y=l} = -\frac{\exp(\mu l)}{l}\left(\frac{\mu}{\sinh \mu l} + 2\sum_{n=1}^{\infty}(-1)^n \frac{\lambda_n^2 \exp[-\nu(\mu^2+\lambda_n^2)t]}{\mu^2+\lambda_n^2}\right).$$

Asymptotic (for large t) representations of the tangential stress vectors are given

$$\hat{f}_0 = \rho\nu\mu(1-\coth \mu l)\hat{u}(t), \qquad \hat{f}_l = -\rho\nu\mu\,\frac{\exp(\mu l)}{\sinh \mu l}\,\hat{u}(t), \qquad \mu = \frac{a}{2\nu}.$$

From these expressions, it follows that the frictional forces exerted on the walls of the slot depend markedly on the fluid crossflow velocity.

The obtained velocity field and tangential stress vectors exerted by the fluid on the plates can be used to take into account forces exerted by fluid flows in channels of various forms, in filtration problems, and in simulation of various physical phenomena in moving fluids.

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